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Kähler manifolds and quantum gravity

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Abstract. The geometry of four-dimensional Kähler manifolds is discussed, and it is shown that the existence of a certain constant spinor enables one to obtain relations between the spectra of wave operators of different spins in an Einstein–Kähler space. This result can be regarded as a generalisation of one discovered recently by Hawking and Pope in spaces with half-flat Riemann tensor.

1. Introduction

There has been considerable interest in recent years in functional integral methods as applied to quantum gravity, quantum field theory on curved space–time backgrounds, and supergravity. In much of this work one begins by 'Wick rotating' to metrics of positive definite signature, in order to improve the convergence properties of the functional integral. In various applications, notably Hawking's space–time foam picture of the gravitational vacuum (Hawking 1978), one is interested in background manifolds which are compact, with no boundaries (i.e. of finite volume), on which one then quantises matter or gravitational fields. At the one-loop level this amounts to calculating the determinants of the second-order operators governing the second variation of the action with respect to the various fields under consideration, in the curved space background which satisfies the classical field equations, i.e. Einstein's equations.

Unless one has examples of such compact manifolds, where an Einstein metric (for which $R_{ab} = \Lambda g_{ab}$) is known explicitly, it is difficult in general to pursue the investigation very far. Explicit examples are in fact uncommon, amounting to S^4, $S^2 \times S^2$, Page's non-trivial S^2 bundle over S^2, $S^1 \times S^1 \times S^1 \times S^1$, and $P_2(\mathbb{C})$ (see, for example, Pope 1981a). One knows however that infinitely many compact Einstein manifolds exist, of arbitrarily great topological complexity.

In an earlier paper (Hawking and Pope 1978a), it was shown how the assumption of self-duality or anti-self-duality of the Riemann tensor ($R_{abcd} = \pm{}^*R_{abcd}$) leads to relations between the determinants of the wave operators for different spins, and in fact means that at the one-loop level the functional integral for supergravity reduces to a finite sum over the zero modes; all the non-zero modes cancel between bosons and fermions. This can be deduced without needing to know the explicit form of the

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metric. Unfortunately K3 is really the only non-trivial compact self-dual space, and so the method is of limited applicability.

The purpose of this paper is to show how in a much wider class of spaces one can deduce a lot about the determinants of the various wave operators without needing to know the metrics explicitly. These spaces are Kähler manifolds, and as will become clear the Einstein–Kähler condition can be regarded as a generalisation of the duality condition $R_{abcd} = \pm {}^*R_{abcd}$. In fact when $\Lambda = 0$ the two conditions become identical. When $\Lambda \neq 0$ there are known to be infinitely many topologically inequivalent Einstein–Kähler manifolds. There are a few isolated cases with $\Lambda > 0$ (of which $S^2 \times S^2$ and $P_2(\mathbb{C})$ are known explicitly, and may in fact be the only ones (e.g. Catenacci and Reina 1979)), but the rest all have $\Lambda < 0$.

The paper concentrates on the basic properties of Kähler and Einstein–Kähler manifolds rather than the applications to specific situations of physical interest. Some of these will be explored further in subsequent work. Much of the material is known already to mathematicians, but is presented here in a manner adapted to the notation of physicists. Section 2 begins by defining a Kähler manifold, and then discusses the exterior algebra of holomorphic and antiholomorphic forms, and shows how in four dimensions this is related to the language of two-component spinors. This relationship depends upon the existence of a certain gauge-covariantly constant spinor, and is really the essential feature of Kähler manifolds on which their special properties depend. In § 3 some topological properties are discussed, and the global question of the existence of spin structures and spin^c structures. Section 4 is concerned with some local properties of the curvature of Kähler manifolds. In § 5 relations are obtained between the eigenfunctions and eigenvalues of various wave operators of different spins in Kähler spaces satisfying the conditions $R =$ constant or $R_{ab} = \Lambda g_{ab}$; these are the analogues of the relations derived by Hawking and Pope (1978a) for half-flat spaces. In § 6 the zero modes of these operators are discussed, and in § 7 the eigenvalue relations are used to evaluate certain zeta functions for different spins. These provide a valuable check on the calculations in earlier sections. In § 8 some possible applications are briefly discussed.

2. Kähler manifolds

We begin by defining an *almost complex manifold* as a real $2n$-dimensional manifold M for which at each point x there exists a two-tensor $\hat{J}$ (with components $J^a{}_b$) which acts as an endomorphism on the tangent space and which is a complex structure for $T_x(M)$; i.e. $\hat{J}^2 = -1$, or in components, $J^a{}_b J^b{}_c = -\delta^a{}_c$. $\hat{J}$ is called the *almost complex structure* of M (Kobayashi and Nomizu 1969).

The *torsion* of $\hat{J}$ is defined by

$$N(X, Y) = 2\{[\hat{J}X, \hat{J}Y] - [X, Y] - \hat{J}[X, \hat{J}Y] - \hat{J}[\hat{J}X, Y]\} \tag{2.1}$$

where X and Y are arbitrary vectors and $\hat{J}X$ denotes the vector with components $(\hat{J}X)^a = J^a{}_b X^b$. If the torsion vanishes then $\hat{J}$ is integrable; it is then called a *complex structure* for M, and M is a *complex manifold.*

A Riemannian metric g on M is said to be *Hermitian* if it is invariant by $\hat{J}$; i.e. $g(\hat{J}X, \hat{J}Y) = g(X, Y)$ for all X and Y, or in component notation, $g_{ab} J^a{}_c J^b{}_d = g_{cd}$. An (almost) complex manifold is said to be an (almost) Hermitian manifold if it admits an Hermitian metric. If the upper index on $\hat{J}$ is lowered using this metric, it follows that the resulting tensor is antisymmetric, and may thus be regarded as a two-form. In a local orthonormal basis e^a, we may therefore define the two-form J by $J = \frac{1}{2}J_{ab}e^a \wedge e^b$.

A *Kähler manifold* is an Hermitian manifold for which the two-form J is closed, $dJ = 0$. J is then called the *Kähler form.* In fact it follows from the previous definitions that $dJ = 0$ implies that J is covariantly constant, $\nabla_a J_{bc} = 0$. J is non-degenerate, and

$$J^n = n!\,\varepsilon \tag{2.2}$$

where J^n denotes the wedge product of n J's, and ε is the volume $2n$-form ($= *1 = e^1 \wedge e^2 \wedge \ldots \wedge e^{2n}$) of M.

Instead of being regarded as a real $2n$-dimensional manifold, M can equivalently be regarded as a complex manifold of dimension n. Thus one can write the metric in the alternative forms

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = e^a \otimes e^a = 2g_{i\bar{j}}\, d\zeta^i\, d\bar{\zeta}^{\bar{j}} = z^m \otimes \bar{z}^m \tag{2.3}$$

where x^μ ($\mu - 1, 2n$) are local real coordinates, e^a ($a = 1, 2n$) is a local orthonormal basis of one-forms, ζ^i ($i = 1, n$) are local complex coordinates and z^m ($m = 1, n$) is a local basis of complex one-forms, which might typically be related to e^a by

$$z^m = e^m + ie^{n+m}, \qquad m = 1, n. \tag{2.4}$$

The way in which the real and the complex descriptions are related is given by the complex structure $\hat{J}$. Since $\hat{J}^2 = -1$, the eigenvalues of $\hat{J}$ are $\pm i$. Vectors for which $\hat{J}X = iX$ or $\hat{J}X = -iX$ will be called *holomorphic* or *antiholomorphic* respectively. By using the metric this definition can be extended to one-forms also. Adopting the natural convention that the one-forms z^m defined by (2.4) are holomorphic, $\hat{J}(z^m) = iz^m$, and so using (2.4) the real components $J^a{}_b$ of $\hat{J}$ may be determined.

Because the basis forms z^m have been chosen to be holomorphic, an arbitrary holomorphic one-form ω may be written as $\omega = \omega_m z^m$, and an antiholomorphic one-form $\eta = \eta_m \bar{z}^m$, so the bundle of one-forms Λ^1 has been divided into two sub-bundles: $\Lambda^{1,0}$ of holomorphic one-forms and $\Lambda^{0,1}$ of antiholomorphic one-forms. Similarly Λ^r, the bundle of r-forms, may be decomposed into sub-bundles $\Lambda^{p,q}$ of type (p, q)-forms $(p+q=r)$, where an arbitrary (p, q)-form α is a sum of wedge products each involving p holomorphic and q antiholomorphic basis one-forms:

$$\alpha = \alpha_{m_1 \ldots m_p n_1 \ldots n_q} z^{m_1} \wedge \ldots \wedge z^{m_p} \wedge \bar{z}^{n_1} \wedge \ldots \wedge \bar{z}^{n_q}. \tag{2.5}$$

The usual exterior derivative d, which maps $\Lambda^r \to \Lambda^{r+1}$, may be decomposed as $d = \partial + \bar{\partial}$, where $\partial : \Lambda^{p,q} \to \Lambda^{p+1,q}$ and $\bar{\partial} : \Lambda^{p,q} \to \Lambda^{p,q+1}$. From $d^2 = 0$ it follows that

$$\partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0. \tag{2.6}$$

The Hodge $*$ operator is defined, as usual, by

$$*(e^{a_1} \wedge \ldots \wedge e^{a_r}) = [(2n-r)!]^{-1} \varepsilon_{a_1 \ldots a_r b_1 \ldots b_{2n-r}} e^{b_1} \wedge \ldots \wedge e^{b_{2n-r}}. \tag{2.7}$$

The $*$ operator maps Λ^r into Λ^{2n-r}, and $\Lambda^{p,q}$ into $\Lambda^{n-q,n-p}$. It is sometimes more convenient to define a new operator $\bar{*}$, by $\bar{*}(\omega) = *\bar{\omega}$ which maps $\Lambda^{p,q}$ into $\Lambda^{n-p,n-q}$. Using this operator the adjoints of d, ∂ and $\bar{\partial}$, may be defined in terms of the Hodge inner product

$$(\alpha, \beta) = \int_M \alpha \wedge \bar{*}\beta \tag{2.8}$$

where $\alpha, \beta \in \Lambda^{p,q}$. Thus by definition

$$(\phi, \mathrm{d}\psi) = (\mathrm{d}^*\phi, \psi) \tag{2.9}$$

and similarly for ∂^* and $\bar\partial^*$, where ϕ and ψ are forms of the appropriate types. Thus

$$\mathrm{d}^* = -\bar{*}\bar{\mathrm{d}}\bar{*} = -*\mathrm{d}*, \qquad \partial^* = -\bar{*}\bar{\partial}\bar{*} = -*\bar{\partial}*, \qquad \bar{\partial}^* = -\bar{*}\bar{\bar{\partial}}\bar{*} = -*\partial*. \tag{2.10}$$

These effect the mappings $\mathrm{d}^*: \Lambda^r \to \Lambda^{r-1}$, $\partial^*: \Lambda^{p,q} \to \Lambda^{p-1,q}$, $\bar\partial^*: \Lambda^{p,q} \to \Lambda^{p,q-1}$.

In this paper we shall be considering specifically the case of Kähler manifolds of four real dimensions, i.e. $n = 2$. This case is special because the $*$ operator carries two-forms into two-forms; and it turns out that the Kähler form has the additional property of having a definite duality, either self-dual ($J = *J$) or anti-self-dual ($J = -*J$). Reversing the orientation of the manifold interchanges the two cases, so without loss of generality we may take the Kähler form to be self-dual.

To conform with standard relativity conventions, coordinate-basis indices $\alpha, \beta, \ldots$ and local tetrad indices $a, b, \ldots$ will be taken to run from 0 to 3 rather than 1 to 4. Where an explicit representation for J is required, we will determine it from the assignment

$$z^1 = e^0 + \mathrm{i}e^3, \qquad z^2 = e^1 + \mathrm{i}e^2, \tag{2.11}$$

rather than (2.4). This then implies that J is given by

$$J = e^0 \wedge e^3 + e^1 \wedge e^2, \tag{2.12}$$

which is indeed self-dual in the convention $\varepsilon_{0123} = +1$.

We now turn to the two-component spinor description of tensors and spinors in Riemannian four-manifolds, and the relationships between this and the complex forms discussed above. It is this connection between the two viewpoints which is really the crucial property of Kähler manifolds to be used in the rest of this paper.

In the Riemannian regime the local tetrad rotation group (i.e. in the tangent space) is SO(4), which is isomorphic to $\mathrm{SU}(2) \times \mathrm{SU}(2)/\mathrm{Z}_2$. Thus instead of regarding vectors (or one-forms) as objects with a single local tetrad index transforming under SO(4), one can instead associate them with objects possessing two indices, one transforming under one of the SU(2) groups and the other under the other. Given a one-form, $V = V_a e^a$, we do this as follows:

$$v = V_{AA'} = (1/\sqrt{2})(V_0 + \mathrm{i}V_i\tau_i), \tag{2.13}$$

where τ_i are the Pauli matrices. Thus

$$v = V_{AA'} = \frac{1}{\sqrt{2}}\begin{pmatrix} V_0 + \mathrm{i}V_3 & V_2 + \mathrm{i}V_1 \\ -V_2 + \mathrm{i}V_1 & V_0 - \mathrm{i}V_3 \end{pmatrix}. \tag{2.14}$$

We choose to regard the *primed* index as labelling rows, and the *unprimed* index as labelling columns. An arbitrary SO(4) rotation on V_a is now expressible in terms of the left and right multiplication of v by certain SU(2) matrices.

To make this isomorphism explicit, we introduce the set of three self-dual generators of SO(4) J^{+i}_{ab} ($i = 1, 2, 3$) and three anti-self-dual generators J^{-i}_{ab} ($*J^{\pm i} = \pm J^{\pm i}$) defined by

$$J^{\pm i}_{0j} = \mp\delta_{ij}, \qquad J^{\pm i}_{jk} = \varepsilon_{ijk}. \tag{2.15}$$

Regarded as 4×4 matrices, they satisfy the quaternion algebra

$$J^{\pm i}J^{\pm j} = -\delta_{ij}\mathbf{1} + \varepsilon_{ijk}J^{\pm k}. \tag{2.16}$$

Writing the tetrad components V_a of V as a column vector

$$V = \begin{pmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{pmatrix}, \tag{2.17}$$

an arbitrary SO(4) rotation U acts on V thus:

$$V' = UV = \exp(R^iJ^{+i})\exp(L^iJ^{-i})V, \tag{2.18}$$

where the three quantities R^i (L^i) are the coefficients of self-dual (anti-self-dual) rotations.

On the other hand, this rotation corresponds to $v' = avb$, where v is the 2×2 complex matrix defined by (2.14), and a and b are the SU(2) matrices:

$$a = \exp(\mathrm{i}L^i\tau_i), \qquad b = \exp(\mathrm{i}R^i\tau_i). \tag{2.19}$$

The Z_2 factor in the isomorphism $SO(4) \cong SU(2)\times SU(2)/Z_2$ is reflected in the fact that a and b can just as well be replaced by $-a$ and $-b$, since SU(2) has Z_2 as centre. Note that the left-hand SU(2) corresponds to anti-self-dual rotations, and the right-hand SU(2) corresponds to self-dual rotations.

As well as 2×2 complex matrices $v = V_{AA'}$ which are isomorphic to vectors, one can consider also row vectors $\phi_A = (\alpha, \beta)$ and column vectors $\psi_{A'} = \binom{\mu}{\nu}$. These are known respectively as right-handed and left-handed spinors: ϕ_A transforms under $SU(2)_R$ and $\psi_{A'}$ under $SU(2)_L$. Unlike the situation in the Lorentzian regime (e.g. Hawking and Pope 1978a), complex conjugation leaves unprimed spinors unprimed and primed spinors primed. There are antisymmetric metrics ε_{AB} for unprimed and $\varepsilon_{A'B'}$ for primed indices. These and their inverses may be used to raise and lower indices, e.g. $\xi_B = \xi^A\varepsilon_{AB}$, etc.

The SU(2) norm $|\phi_A|^2$ may be defined by means of complex conjugation. When a spinor is complex conjugated a lower index becomes raised and *vice versa*, so one has

$$|\phi_A|^2 = (\overline{\phi_A})\phi_A = \bar{\phi}^A\phi_A \geqslant 0, \tag{2.20}$$

equality implying $\phi_A = 0$. A positive definite norm for left-handed spinors may be defined in the same way.

The usual Lorentzian signature two-component spinor conventions use a metric of signature $(+\ -\ -\ -)$ which 'Riemannianises' to a metric of negative definite signature. In order to have a positive definite signature it is necessary to introduce certain minus signs into the usual conventions. In terms of the tetrad components used in (2.14) the Van de Waerden symbols $\sigma^a{}_{AA'}$ which relate V_a to $V_{AA'} = V_a\sigma^a{}_{AA'}$ are given by

$$\sigma^a{}_{AA'} = (1/\sqrt{2})(1, \mathrm{i}\boldsymbol{\tau}). \tag{2.21}$$

The orthonormal metric δ_{ab} is given in terms of σ by

$$\delta_{ab} = -\sigma_a{}^{AA'}\sigma_b{}^{BB'}\varepsilon_{AB}\varepsilon_{A'B'}. \tag{2.22}$$

The minus sign in (2.22) means that one has to be slightly careful when transcribing from tensor to spinor notation; for example

$$V^a W_a = -V^{AA'} W_{AA'}.$$

In the standard manner, one can introduce arbitrary multi-index spinors $\phi_{A_1\ldots A_{2m}A'_1\ldots A'_{2n}}$ with $2m$ unprimed and $2n$ primed indices. If ϕ is totally symmetrised over unprimed indices and also over primed indices, then it is in the $(2m+1)(2n+1)$-dimensional (n, m) irreducible representation of SO(4). Thus left- and right-hand spinors are respectively in the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations, and vectors are in the $(\frac{1}{2}, \frac{1}{2})$ representation. The cases $(1, 0)$ and $(0, 1)$ correspond respectively to anti-self-dual and self-dual two-forms; i.e. anti-self-dual two-forms are represented by symmetric two-index primed spinors $\tilde{\phi}_{A'B'}$ and self-dual two-forms by symmetric unprimed spinors ϕ_{AB}. Adopting the usual convention in which the presence of Van der Waerden symbols is understood, one can write an arbitrary two-form $\Phi = \frac{1}{2}\Phi_{ab}e^a \wedge e^b$ as the sum of its self-dual and anti-self-dual parts:

$$\Phi_{ab} = \tfrac{1}{2}\phi_{AB}\varepsilon_{A'B'} + \tfrac{1}{2}\tilde{\phi}_{A'B'}\varepsilon_{AB}. \tag{2.23}$$

In order to investigate the relationship between holomorphic or antiholomorphic forms and two-component spinors, we first need to establish the existence of a certain right-handed spinor u_A, which has the property of being gauge-covariantly constant, $D_a u_A = 0$, where $D_a = \nabla_a - ieA_a$, and A is a connection on a certain U(1) bundle. The object $\sigma^a{}_{AA'}\bar{u}^A$ then maps between left-handed spinors and one-forms, and it will turn out that the one-forms are antiholomorphic, i.e. elements of $\Lambda^{0,1}$. It will also turn out that right-hand spinors are associated with certain elements of $\Lambda^{0,0}$ and $\Lambda^{0,2}$.

To begin with, we will consider an arbitrary four-dimensional Riemannian space and introduce another covariant derivative, $\mathscr{D}_a$, defined as follows. The self-dual generators $J^{\pm i}_{ab}$ are not, in general, covariantly constant; but one can easily show that

$$\nabla_c J^{+i}_{ab} + \varepsilon_{ijk} A^{+j}_c J^{+k}_{ab} = 0, \tag{2.24}$$

where

$$A^{+i} = \tfrac{1}{2} J^{+i}_{ab}\omega_{ab}, \tag{2.25}$$

ω_{ab} being the connection one-forms defined by $de^a = -\omega^a{}_b \wedge e^b$, $\omega_{(ab)} = 0$. Equation (2.24) is therefore just the statement that $J^{+i}{}_{ab}$ is gauge-covariantly constant with respect to the derivative $\mathscr{D}_a$ which involves a minimally coupled SU(2) Yang–Mills gauge field with connection (2.25); $\mathscr{D}_c J^{+i}_{ab} = 0$. The curvature of this connection (i.e. the Yang–Mills field strength) is easily shown to be

$$F^{+i} = \tfrac{1}{2} J^{+i}_{ab}\Theta_{ab}, \tag{2.26}$$

where Θ_{ab} are the curvature two-forms defined by $\Theta_{ab} = d\omega_{ab} + \omega_{ac} \wedge \omega_{cb}$. One can of course do the same thing with the anti-self-dual generators J^{-i}_{ab}. It is interesting to note that (2.26) and the anti-self-dual analogue are just the Yang–Mills fields discussed by Charap and Duff (1977); they are nothing more than the curvatures of the right- and left-hand spin bundles.

The next stage in the argument is to look at the holonomy group for the general four-manifold, by considering the parallel propagation of vectors, and then spinors, around closed loops. If one parallel propagates V^a around a small loop of area $\delta\Sigma^{ab}$, then as is well known,

$$\delta V^a = -R^a{}_{bcd}\delta\Sigma^{cd} V^b. \tag{2.27}$$

This may also be written as

$$\delta V^a = (\delta R^i J^{+i}_{ab} + \delta L^i J^{-i}_{ab}) V^b, \tag{2.28}$$

since clearly the final vector is just an SO(4) rotation of the initial one. Straightforward algebra then shows that

$$\delta R^i = -\tfrac{1}{2} F^{+i}_{ab} \delta\Sigma^{ab}, \qquad \delta L^i = -\tfrac{1}{2} F^{-i}_{ab} \delta\Sigma^{ab}. \tag{2.29}$$

Thus for a loop of any size spanned by a two-surface C, one can write

$$V' = \exp(R^i J^{+i}) \exp(L^i J^{-i}) V, \tag{2.30}$$

where we are now using the matrix notation of equation (2.18). As discussed previously, a right-handed spinor ϕ (a row vector) and a left-handed spinor ψ (a column vector) will suffer rotations

$$\phi' = \phi \exp(\mathrm{i} R^i \tau^i), \qquad \psi' = \exp(\mathrm{i} L^i \tau^i) \psi. \tag{2.31}$$

Because the curvatures $F^{\pm}$ take values in a non-abelian group, one cannot simply integrate (2.29) to give

$$R^i = -\tfrac{1}{2} \int_C F^{+i}, \qquad L^i = -\tfrac{1}{2} \int_C F^{-i}. \tag{2.32}$$

However it will turn out that in the case of a Kähler manifold the right-hand spin bundle is a U(1) bundle rather than SU(2), for which case equation (2.32) for R^i is valid.

We now come to the crucial property of Kähler spaces: the self-dual part of the SO(4) rotation, $R^i J^{+i}$, instead of generating $SU(2)_R$ rotations, in fact collapses down to just θJ where θ is a single parameter and J is the Kähler form (self-dual). In other words, right-handed spinors suffer only U(1) rotations when parallel transported around closed curves.

To see this, define the isovector scalar ϕ^i by

$$\phi^i = \tfrac{1}{4} J^{+1}_{ab} J_{ab}. \tag{2.33}$$

This satisfies $\phi^i \phi^i = 1$, and since J_{ab} is covariantly constant, and J^{+i}_{ab} is gauge-covariantly constant with respect to $\mathscr{D}_a$, it follows that $\mathscr{D}_a \phi^i = 0$. Taking a commutator of second derivatives then implies $F^{+i}_{ab} \phi^j \varepsilon_{ijk} = 0$ and hence $F^{+i}_{ab} = \phi^i P_{ab}$ for some P_{ab}. Inverting this relation shows that

$$P_{ab} = \tfrac{1}{2} R_{abcd} J_{cd}, \tag{2.34}$$

a two-form known as the Ricci form in a Kähler manifold.

One can in fact choose a gauge in which $R^1 = R^2 = 0$, and so equation (2.29) for R^i can now be integrated to give

$$R^i J^{+i} = R^3 J^{+3} = \theta J, \qquad \theta = -\tfrac{1}{2} \int_C P, \tag{2.35}$$

where $P = \frac{1}{2} P_{ab} e^a \wedge e^b$ and C is any two-surface spanning the closed path. From (2.31) a right-handed spinor ϕ suffers a rotation

$$\phi' = \phi\, \mathrm{e}^{-\mathrm{i}\theta\tau_3} = \phi \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\theta} & \\ & \mathrm{e}^{\mathrm{i}\theta} \end{pmatrix}. \tag{2.36}$$

Thus a right-handed spinor is rotated by a U(1) subgroup of the $SU(2)_R$ factor in

$SO(4) = SU(2)_L \times SU(2)_R/Z_2$. In fact, this shows that the holonomy group for a four real dimensional Kähler manifold is $SU(2) \times U(1)/Z_2 \cong U(2)$, which is an alternative way of defining a Kähler manifold (Kobayashi and Nomizu 1969).

In order to find a gauge-covariant constant spinor u_A it is necessary to find some way of 'undoing' the rotation of equation (2.36) so that the spinor is left unrotated after parallel propagation around all possible closed loops. The fact that the rotation (2.36) is just a U(1) subgroup of SU(2) immediately suggests how this might be achieved; by giving u_A an electric charge and minimally coupling it to some suitable U(1) gauge field (i.e. Maxwell field). Let us therefore introduce the covariant derivative $D_a = \nabla_a - ieA_a$, with $dA = F$, and look for an F which achieves the required result.

The effect of gauge-parallel propagating the spinor ϕ around a closed loop in the presence of this U(1) gauge field will be to introduce an extra phase factor into (2.36), giving

$$\phi' = \phi\begin{pmatrix} e^{-i\theta} & \\ & e^{i\theta} \end{pmatrix} \exp\left(ie\int_C F\right), \tag{2.37}$$

where as before, $\theta = -\frac{1}{2}\int_C P$, and P is the Ricci form (2.34). Clearly therefore if we choose F so that $\int_C P = 2e\int_C F$ for all paths, then any right-handed spinor $\phi = (0, \beta)$ will be unrotated, or if we choose $\int_C P = -2e\int_C F$, then any right-handed spinor $\phi = (\alpha, 0)$ will be unrotated. But this is just the condition

$$F_{ab} = \pm(1/2e)P_{ab}, \tag{2.38}$$

so by gauge coupling ϕ to a suitable multiple of the Ricci form, a gauge-covariantly constant spinor can be found. Since the norm (equation (2.20)) of such a spinor must be constant, that means that α and β must be constants, so the spinor may without loss of generality be taken to be $u_A = (0, 1)$ or $u_A = (1, 0)$ respectively, depending on the choice of sign in (2.38).

Choosing the case $F = +(1/2e)P$, so $u_A = (0, 1)$, it follows that $\bar{u}_A = (-1, 0)$, so as one would expect $\bar{u}_A$ is gauge-covariantly constant also, with charge $-e$. One can now show by using the Van der Waerden symbols (2.21) to transcribe the Kähler form J (see equation (2.12)) from spinor form to tensor form, $J_{ab} = J_{AA'BB'} = \frac{1}{2}J_{AB}\varepsilon_{A'B'}$, that it may be written as

$$J_{AB} = 2i(u_A\bar{u}_B + u_B\bar{u}_A). \tag{2.39}$$

(Being self-dual, the spinor transcription of J_{ab} involves only J_{AB}, and not $J_{A'B'}$, see equation (2.23).) It is immediately clear from (2.39) that J is covariantly constant, since it is made from products of two gauge-covariantly constant spinors, one having charge $+e$ and the other $-e$.

The vector space of right-handed spinors is two dimensional, and so is spanned by the pair of spinors $(u_A, \bar{u}_A)$, which may therefore be taken as a basis for the space. The antisymmetric metric ε_{AB} may be written as

$$\varepsilon_{AB} = u_A\bar{u}_B - u_B\bar{u}_A. \tag{2.40}$$

We are now in a position to discuss the isomorphism between spinors and antiholomorphic forms (the fact that the forms are antiholomorphic rather than holomorphic is the result of convention only, and is chosen to agree with the standard literature, e.g. Wells (1979)). Consider an arbitrary left-handed spinor $\psi^{A'}$, and convert it into a one-form ω by contracting it into $\sigma_{aAA'}\bar{u}^A$:

$$\omega_a = \sigma_{aAA'}\bar{u}^A\psi^{A'}, \qquad \text{or} \qquad \omega_{AA'} = \bar{u}_A\psi_{A'}. \tag{2.41}$$

Remembering that one-forms for which $J(\omega) = +\mathrm{i}\omega$ or $-\mathrm{i}\omega$ are respectively holomorphic or antiholomorphic, where $(J\omega)_a = J_a{}^b\omega_b$, we evaluate $J(\omega)$ and find

$$J(\omega)_{AA'} = -\mathrm{i}\bar{u}_A\psi_{A'} = -\mathrm{i}\omega_{AA'}, \tag{2.42}$$

so ω defined by (2.41) is an antiholomorphic one-form, $\omega \in \Lambda^{0,1}$ (clearly if u^A were used in (2.41) rather than $\bar{u}^A$, ω would be holomorphic).

Turning now to a right-handed spinor ϕ_A, there are two ways of using $\bar{u}_A$ to associate antiholomorphic forms with ϕ_A. One is to contract the index, giving a zero-form $\bar{u}^A\phi_A$. The other is to construct the self-dual two-form ν with spinor transcription $\bar{u}_A\phi_B + \bar{u}_B\phi_A$. Now ϕ_A may be expanded as $\phi_A = au_A + b\bar{u}_A$, so this two-form is given by $\nu_{ab} = \frac{1}{2}\nu_{AB}\varepsilon_{A'B'}$, where

$$\nu_{AB} = a(\bar{u}_Au_B + \bar{u}_Bu_A) + 2b\bar{u}_A\bar{u}_B = -\tfrac{1}{2}\mathrm{i}aJ_{AB} + 2b\bar{u}_A\bar{u}_B. \tag{2.43}$$

In view of (2.42), which showed that an unprimed index on $\bar{u}_A$ together with a primed index on any spinor constitutes an antiholomorphic one-form index (holomorphic if u_A is used instead of $\bar{u}_A$), we see that the term in ν involving aJ_{AB} is an element of $\Lambda^{1,1}$, and the term involving $b\bar{u}_A\bar{u}_B$ is an element of $\Lambda^{0,2}$. But $b = -u^A\phi_A$, so in other words there is a natural isomorphism between a right-handed spinor ϕ_A and the element $\bar{u}^A\phi_A$ of $\Lambda^{0,0}$ which contains the $\bar{u}^A$ projection of ϕ_A, and the element $-u^C\phi_C\bar{u}_A\bar{u}_B\varepsilon_{A'B'}$ of $\Lambda^{0,2}$ which contains the u^A projection of ϕ_A. Thus denoting the left- and right-hand spin bundles by S^- and S^+ respectively, we have

$$S^- \cong \Lambda^{0,1} \equiv \Lambda^{\mathrm{odd}}, \qquad S^+ \cong \Lambda^{0,0} \oplus \Lambda^{0,2} \equiv \Lambda^{\mathrm{even}}. \tag{2.44}$$

It is now apparent that there is at least a formal similarity between the Dirac operator, which interchanges the right- and left-handed spin bundles, and the antiholomorphic exterior derivative $\bar{\partial}$ and its adjoint $\bar{\partial}^*$, which interchange Λ^{odd} and Λ^{even}. In fact, a straightforward calculation shows that the charged Dirac operator, which acts on $\phi_A \in S^+$ or $\psi_{A'} \in S^-$ by

$$D_{AA'}\phi^A \in S^-, \qquad D_{AA'}\psi^{A'} \in S^+, \tag{2.45}$$

is isomorphic to the operator $\sqrt{2}(\bar{\partial} + \bar{\partial}^*)$ acting on the direct sum of $\Lambda^{0,q}$ forms $= \Lambda^{\mathrm{odd}} \oplus \Lambda^{\mathrm{even}}$ (Hitchin 1974). This may be expressed in the following commutative diagram:

$$\begin{array}{ccc} S^- & \cong & \Lambda^{\mathrm{odd}} \\ {\scriptstyle D_{AA'}}\updownarrow & & \updownarrow{\scriptstyle \sqrt{2}(\bar{\partial}+\bar{\partial}^*)} \\ S^+ & \cong & \Lambda^{\mathrm{even}} \end{array} \tag{2.46}$$

The operators $\bar{\partial}$ and $\bar{\partial}^*$ are now in general taken to act on forms which are charged, and so for reasons of clarity we introduce three gauged exterior derivative operators D, D^+ and D^- to generalise d, ∂ and $\bar{\partial}$ respectively, where

$$\mathrm{D} = \mathrm{D}^+ + \mathrm{D}^- = \mathrm{d} - \mathrm{i}enA, \qquad \mathrm{D}^+ = \partial - \mathrm{i}enA^+, \qquad \mathrm{D}^- = \bar{\partial} - \mathrm{i}enA^-, \tag{2.47}$$

and A^+ and A^- are the holomorphic and antiholomorphic parts of the connection A on the U(1) fibres, and ne is the charge of the forms on which the derivatives are acting. Thus $\mathrm{D}^+ : \Lambda^{p,q} \to \Lambda^{p+1,q}$, $\mathrm{D}^- : \Lambda^{p,q} \to \Lambda^{p,q+1}$. In the case of uncharged forms, these new derivatives reduce to the previous ones. In terms of these, the operator isomorphic to the charged Dirac operator is $\sqrt{2}(\mathrm{D}^- + \mathrm{D}^{-*})$.

In § 5 we will make use of the results discussed above to obtain relations between the eigenfunctions of the Hodge–de Rham Laplacians in different irreducible representations of SO(4), i.e. for wave operators of different spins (see, for example, Christensen and Duff 1979), and show how the various eigenvalue spectra are related in an Einstein–Kähler background.

We conclude this section by investigating the structure of self-dual (Λ^2_+) and anti-self-dual (Λ^2_-) two-forms in terms of the three elements in the direct sum $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_- = \Lambda^{2,0} \oplus \Lambda^{1,1} \oplus \Lambda^{0,2}$. Beginning with anti-self-dual two-forms, these have the spinor transcription $\frac{1}{2}\tilde{\phi}_{A'B'}\varepsilon_{AB}$ for some $\tilde{\phi}_{A'B'}$ (see equation (2.23)), and so by virtue of (2.40), (2.41) and (2.42), all such two-forms have one holomorphic and one antiholomorphic index, and so lie in $\Lambda^{1,1}$. Self-dual two-forms have the spinor transcription $\frac{1}{2}\phi_{AB}\varepsilon_{A'B'}$, which can be rewritten as $(\alpha u_A u_B + \beta \bar{u}_A \bar{u}_B + \gamma u_{(A}\bar{u}_{B)})\varepsilon_{A'B'}$, since $u_A u_B$, $\bar{u}_A \bar{u}_B$ and $u_{(A}\bar{u}_{B)}$ span the space of symmetric two-index unprimed spinors. The first term clearly lies in $\Lambda^{2,0}$, the second in $\Lambda^{0,2}$ and the third, which is proportional to $J_{AB}\varepsilon_{A'B'}$, lies in $\Lambda^{1,1}$. Thus we can write

$$\Lambda^2_- = \Lambda^{1,1}_\perp, \qquad \Lambda^2_+ = \Lambda^{2,0} \oplus \Lambda^{0,2} \oplus \Lambda^{1,1}_J, \tag{2.48}$$

where $\Lambda^{1,1}_J$ denotes the subspace of $\Lambda^{1,1}$ proportional to J, the Kähler form, and $\Lambda^{1,1}_\perp$ denotes the orthogonal complement: $\Lambda^{1,1} = \Lambda^{1,1}_J \oplus \Lambda^{1,1}_\perp$.

The two-forms whose spinor equivalents are proportional to $u_A u_B$ and $\bar{u}_A \bar{u}_B$ will be used repeatedly in later sections, so we will make the following definition. Let $K = \frac{1}{2}K_{ab}e^a \wedge e^b$ be the following self-dual two-form:

$$K_{ab} = \tfrac{1}{2}K_{AB}\varepsilon_{A'B'}, \qquad K_{AB} = 4u_A u_B, \tag{2.49}$$

and let $L = \bar{K}$, the complex conjugate of K. Converting into tensor notation by using the Van de Waerden symbols (2.21), one finds

$$\begin{aligned} K &= \mathrm{i}(e^0 \wedge e^1 + e^2 \wedge e^3) - (e^0 \wedge e^2 + e^3 \wedge e^1) = \mathrm{i}z^1 \wedge z^2, \\ L &= -\mathrm{i}(e^0 \wedge e^1 + e^2 \wedge e^3) - (e^0 \wedge e^2 + e^3 \wedge e^1) = -\mathrm{i}\bar{z}^1 \wedge \bar{z}^2, \end{aligned} \tag{2.50}$$

so that as expected, $K \in \Lambda^{2,0}$ and $L \in \Lambda^{0,2}$. For completeness, we also note that in the complex basis (z^1, z^2), the Kähler form J (see equation (2.7)) is given by

$$J = \tfrac{1}{2}\mathrm{i}(z^1 \wedge \bar{z}^1 + z^2 \wedge \bar{z}^2) \tag{2.51}$$

3. Some topological properties of four-dimensional Kähler manifolds

We begin by recalling some facts about the Chern classes of a complex vector bundle. A more detailed discussion intended for physicists can be found in the excellent review article by Eguchi *et al* (1980). The Chern form of a complex vector bundle E with connection Γ over a manifold M is defined as

$$c(\Omega) = \det[I + (\mathrm{i}/2\pi)\Omega] = 1 + c_1(\Omega) + c_2(\Omega) + \ldots, \tag{3.1}$$

where Ω is the curvature two-form of the bundle, and the Chern forms $c_j(\Omega)$ are polynomials of degree j in Ω. In the case of four dimensions the series terminates with c_2, and one has

$$c_1 = (\mathrm{i}/2\pi)\,\mathrm{Tr}\,\Omega, \tag{3.2}$$

$$c_2 = (8\pi^2)^{-1}(\mathrm{Tr}\,\Omega \wedge \Omega - \mathrm{Tr}\,\Omega \wedge \mathrm{Tr}\,\Omega). \tag{3.3}$$

The Chern forms $c_j(\Omega)$ are closed, $\mathrm{d}c_j(\Omega) = 0$, and define $2j$th cohomology classes. These classes are in fact integer classes $H^{2j}(M, \mathbb{Z})$, so if $c_j(\Omega)$ is integrated over any $2j$ cycle, the result is an integer.

In one integrates $c_1 \wedge c_1$, or c_2, over M, then one obtains the first or second Chern numbers, C_1^2 or C_2, which are integer topological invariants,

$$C_1^2 = \int_M c_1 \wedge c_1, \qquad C_2 = \int_M c_2. \tag{3.4}$$

One defines the Chern classes of a complex manifold to be the Chern classes of its complex tangent space, which in a Kähler manifold means that the matrix of curvature two-forms Ω in (3.1) is that obtained from the usual curvature two-forms Θ_{ab} by projecting out the holomorphic part of one index, and the antiholomorphic part of the other:

$$\Omega_{ab} = \tfrac{1}{4}(\delta_{ac} + \mathrm{i}J_{ac})(\delta_{bd} - \mathrm{i}J_{bd})\Theta_{cd}. \tag{3.5}$$

Hence $\mathrm{Tr}\,\Omega = \frac{1}{2}\mathrm{i}J_{ab}\Theta_{ab} = -\mathrm{i}P$ (see (2.34)), where P is the Ricci form, and $\mathrm{Tr}\,\Omega \wedge \Omega = \frac{1}{2}\mathrm{Tr}\Theta \wedge \Theta$. Therefore c_1 and c_2 are given by

$$c_1 = \frac{1}{2\pi}P, \qquad c_2 = \frac{1}{16\pi^2}\mathrm{Tr}\,\Theta \wedge \Theta + \frac{1}{8\pi^2}P \wedge P. \tag{3.6}$$

From (3.4), one therefore has

$$C_1^2 = \frac{1}{4\pi^2}\int_M P \wedge P, \qquad C_2 = \frac{1}{16\pi^2}\int_M \mathrm{Tr}\,\Theta \wedge \Theta + \tfrac{1}{2}C_1^2. \tag{3.7}$$

But C_2, being the highest Chern number in four real dimensions, is equal to the Euler number χ, and $\int \mathrm{Tr}\,\Theta \wedge \Theta$ is proportional to the Hirzebruch signature, τ, which is also a topological invariant:

$$\tau = -\frac{1}{24\pi^2}\int_M \mathrm{Tr}\,\Theta \wedge \Theta, \tag{3.8}$$

so one has the relation

$$2\chi + 3\tau = C_1^2 \tag{3.9}$$

which holds in any Kähler manifold of dimension four.

The Euler number, χ, of a four-manifold can be defined as the alternating sum of the Betti numbers b_r:

$$\chi = \sum_{r=0}^{4} (-1)^r b_r, \tag{3.10}$$

where b_r is the dimension of the rth cohomology class $H^r(M, \mathbb{R})$, i.e. the number of independent closed but inexact r-forms ($\mathrm{d}\omega = 0$, $\omega \neq \mathrm{d}\alpha$). The cohomology class in the middle dimension, i.e. $H^2(M, \mathbb{R})$ in four dimensions, may be split into its self-dual and anti-self-dual parts, and defining the dimensions of these to be b_2^+ and b_2^-

respectively ($b_2^+ + b_2^- = b_2$), the Hirzebruch signature is

$$\tau = b_2^+ - b_2^-. \tag{3.11}$$

The Euler number may be shown, in the case of a compact four-manifold without boundary, to be given by

$$\chi = \frac{1}{32\pi^2} \int_M \varepsilon_{abcd} \Theta_{ab} \wedge \Theta_{cd}, \tag{3.12}$$

and the Hirzebruch signature is given by equation (3.8).

Because in a Kähler manifold the bundle of r-forms Λ^r splits as the direct sum $\Lambda^r = \Sigma_{p+q=r} \Lambda^{p,q}$, the cohomology classes $H^r(M, \mathbb{R})$ can be subdivided into $H^{p,q}(M, \mathbb{R})$, $p+q=r$, $H^{p,q}(M, \mathbb{R})$ having dimension $h^{p,q}$, so $b_r = \Sigma_{p+q=r} h^{p,q}$. One can easily show that in terms of these,

$$\chi = \sum_{p,q} (-1)^{p+q} h^{p,q}, \qquad \tau = \sum_{p,q} (-1)^q h^{p,q}. \tag{3.13}$$

There is another topological invariant which may be defined, namely the *arithmetic genus* a:

$$a = \sum_q h^{0,q} (-1)^q. \tag{3.14}$$

It follows by complex conjugation that $h^{p,q} = h^{q,p}$, and by Hodge duality that $h^{p,q} = h^{2-q,2-p}$, and using these relations it is easy to see from (3.13) and (3.14) that

$$a = \tfrac{1}{4}(\chi + \tau) = \tfrac{1}{12}(\chi + C_1^2). \tag{3.15}$$

Writing (3.14) as $a = h^{0,0} + h^{0,2} - h^{0,1}$, one sees that it is just $h^{\text{even}} - h^{\text{odd}}$ in the language of (2.44), and hence by virtue of the isomorphism between spinors and antiholomorphic forms, a is also the index of the charged Dirac operator $D_{AA'}$; the excess of right-handed over left-handed normalisable solutions of the Dirac equation.

That the arithmetic genus is equal to the Dirac index may also be seen directly, by using the Atiyah–Singer index theorem (e.g. Eguchi *et al* 1980). For the charged Dirac operator, this states that

$$n_+ - n_- = \frac{1}{192\pi^2} \int_M \operatorname{Tr} \Theta \wedge \Theta + \frac{e^2}{8\pi^2} \int_M F \wedge F, \tag{3.16}$$

where n_+ and n_- are the numbers of right- and left-handed square integrable solutions of the charged Dirac equation. Substituting $F = (1/2e)P$ and using (3.7), (3.8) and (3.9), one recovers the result that $n_+ - n_- = \frac{1}{4}(\chi + \tau)$, which by (3.15) is the arithmetic genus.

So far in the discussion of spinors we have not considered the global question of whether the manifold M admits a *spin structure*; i.e. of whether it is possible to define spinors globally. The reason in general why a manifold might fail to admit a spin structure is related to the fact that in N dimensions the group Spin(N) of local spinor rotations is the double cover of SO(N), the group of local tangent space rotations. In the case of four dimensions, $\mathrm{Spin}(4) \cong \mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R}$, where as discussed previously, $\mathrm{SU}(2)_\mathrm{L}$ and $\mathrm{SU}(2)_\mathrm{R}$ act on left-handed and right-handed spinors respectively.

If there is a closed two-surface Y in M which cannot be contracted to zero, then when a vector is parallelly propagated around a one-parameter family of closed curves spanning Y, the two vectors obtained by propagating around the two trivial curves in the family are identical. On the other hand, when the same thing is done for spinors, the two spinors may either be identical, in which case the manifold admits a spin structure, or they may differ by a factor of -1, in which case it does not admit a spin structure. In a four-dimensional Kähler manifold we may look at this in detail by returning to equation (2.36), which gives the rotation which a right-handed spinor ϕ (a two-element row vector) undergoes when parallel-transported around a closed curve spanned by a two-surface C. Thus the two spinors ϕ_1 and ϕ_2 obtained by parallel propagation around the two trivial curves in a family spanning a closed two-surface Y are related by

$$\phi_2=\phi_1\begin{pmatrix} e^{-i\theta} & \\ & e^{i\theta}\end{pmatrix}; \qquad \theta=-\tfrac{1}{2}\int_Y P. \tag{3.17}$$

However, by equation (3.6), $\theta=-\pi\int_Y c_1$, where c_1 is the first Chern class of the complex tangent bundle. Since c_1 defines an integer cohomology class $H^2(M,\mathbb{Z})$, its integral over any two-cycle Y gives an integer, and so it follows from (3.17) that either $\int_Y c_1=$ even integer and $\phi_2=\phi_1$, or $\int_Y c_1=$ odd integer and $\phi_2=-\phi_1$. Thus the condition for the manifold to admit a spin structure is that $\int_Y c_1=$ even integer. In fact the mod 2 reduction of the first Chern class is the second Stiefel–Whitney class W_2, so M admits a spin structure if and only if W_2 vanishes. Thus we have reproduced a well known result of differential geometry (e.g. Eguchi *et al* 1980). (It follows of course, since the product of a left-handed and a right-handed spinor gives a vector—which can always be globally defined—that the existence or non-existence of a right-handed spin structure implies the same for left-handed spinors.)

The foregoing discussion was for the case of uncharged spinors. If we now consider instead the charged spinors introduced in § 2, coupled to the electromagnetic field $F=(1/2e)/P$, the situation is very different. In this case it follows from equation (2.37) that the two spinors ϕ_1 and ϕ_2 will be related by

$$\phi_2=\phi_1\begin{pmatrix}\exp(2\pi i\int c_1) & 0\\ 0 & 1\end{pmatrix} \tag{3.18}$$

and so $\phi_1=\phi_2$ regardless of whether $\int_Y c_1$ is an even or odd integer. What has happened is that in the case where $\int_Y c_1$ is odd, the electromagnetic field has introduced a phase factor of -1 in order to restore the equality of ϕ_1 and ϕ_2. This is known as a *spin^c structure*, and we see from (3.18) that such a structure always exists in a four-dimensional Kähler manifold.

4. The curvature of Kähler four-manifolds

The fact that the holonomy group for right-handed spinors is U(1) rather than SU(2) shows that the right-handed curvature of a four-dimensional Kähler manifold has a special form. In order to discuss this it is convenient to use the two-component spinor description of the curvature. To do this, we first break up the Riemann tensor into its SO(4) irreducible parts: the Weyl tensor C_{abcd}, the tracefree Ricci tensor E_{ab} and the Ricci scalar R:

$$R_{a\ c}^{\ b\ d}=C_{a\ c}^{\ b\ d}+2E_{[a}^{\ \ [b}\delta_{c]}^{\ \ d]}+\tfrac{1}{6}R\delta_{[a}^{\ \ [b}\delta_{c]}^{\ \ d]}, \tag{4.1}$$

where $E_{ab}=R_{ab}-\frac{1}{4}Rg_{ab}$. The Weyl tensor may be divided into its self-dual and anti-self-dual parts, $C_{abcd} = {}^+C_{abcd} + {}^-C_{abcd}$, where ${}^\pm C_{abcd} = \frac{1}{2}(C_{abcd} \pm {}^*C_{abcd})$, ${}^*C_{abcd} = \frac{1}{2}\varepsilon_{abef}C_{efcd}$. The spinor transcriptions of these tensors are

$$ {}^+C_{abcd} = \Psi_{ABCD}\varepsilon_{A'B'}\varepsilon_{C'D'}, \qquad \Psi_{ABCD} = \Psi_{(ABCD)}, \tag{4.2}$$

$$ {}^-C_{abcd} = \tilde{\Psi}_{A'B'C'D'}\varepsilon_{AB}\varepsilon_{CD}, \qquad \tilde{\Psi}_{A'B'C'D'} = \tilde{\Psi}_{(A'B'C'D')}, \tag{4.3}$$

$$ E_{ab} = 2\Phi_{ABA'B'}, \qquad \Phi_{ABA'B'} = \Phi_{(AB)(A'B')}. \tag{4.4}$$

The conventions are such that the commutator $[\nabla_{AA'}, \nabla_{BB'}]$ applied to arbitrary spinors ξ_C, $\tilde{\xi}_{C'}$ gives

$$[\nabla_{AA'}, \nabla_{BB'}]\xi_C = (\Psi_{ABCD}\xi^D - \tfrac{1}{12}R\xi_{(A}\varepsilon_{B)C})\varepsilon_{A'B'} + \Phi_{CDA'B'}\xi^D\varepsilon_{AB}, \tag{4.5}$$

$$[\nabla_{AA'}, \nabla_{BB'}]\tilde{\xi}_{C'} = (\tilde{\Psi}_{A'B'C'D'}\tilde{\xi}^{D'} - \tfrac{1}{12}R\tilde{\xi}_{(A'}\varepsilon_{B')C'})\varepsilon_{AB} + \Phi_{ABC'D'}\tilde{\xi}^{D'}\varepsilon_{A'B'}. \tag{4.6}$$

The Einstein condition $R_{ab} = \Lambda g_{ab}$ is equivalent to $E_{ab} = 0$, or $\Phi_{ABA'B'} = 0$, with $R = 4\Lambda$. The half-flat condition, $R_{abcd} = \pm{}^*R_{abcd}$, which automatically implies $R_{ab} = 0$ (or equivalently $E_{ab} = 0 = R$), corresponds to $\tilde{\Psi}_{A'B'C'D'} = 0$ if R_{abcd} is self-dual, or $\Psi_{ABCD} = 0$ if R_{abcd} is anti-self-dual.

By applying the commutator $[\nabla_{AA'}, \nabla_{BB'}]$ to the covariantly constant Kähler form J_{CD}, one can show that the right-handed Weyl tensor in a Kähler manifold is given by

$$\Psi_{ABCD} = \tfrac{1}{32}RJ_{(AB}J_{CD)} = -\tfrac{1}{2}Ru_{(A}u_B\bar{u}_C\bar{u}_{D)}, \tag{4.7}$$

where the second equality follows from (2.39). This shows that the right-handed Weyl tensor is of Petrov type D. The Kähler condition implies no restriction on the left-handed Weyl tensor. In tensor notation, the right-handed Weyl tensor takes the form

$$ {}^+C_{abcd} = \tfrac{1}{24}R(\Sigma_{abcd} - G_{abcd}), \tag{4.8}$$

where G and Σ are defined by

$$G_{abcd} = g_{ac}g_{bd} - g_{ad}g_{bc}, \qquad \Sigma_{abcd} = J_{ac}J_{bd} - J_{ad}J_{bc} + 2J_{ab}J_{cd}. \tag{4.9}$$

The expression $[\nabla_a, \nabla_b]J_{cd} = 0$ implies that the Riemann tensor satisfies the relation

$$R_{abcd} = R_{abef}J_{ec}J_{fd}, \tag{4.10}$$

from which it follows that the Ricci form P_{ab}, defined by (2.34), may also be written as

$$P_{ab} = R_{ac}J_{cb}. \tag{4.11}$$

In spinor notation, this becomes

$$\begin{aligned} P_{ab} &= \tfrac{1}{2}P_{AB}\varepsilon_{A'B'} + \tfrac{1}{2}\tilde{P}_{A'B'}\varepsilon_{AB}, \\ P_{AB} &= \tfrac{1}{4}RJ_{AB}, \qquad \tilde{P}_{A'B'} = \Phi_{ABA'B'}J^{AB}. \end{aligned} \tag{4.12}$$

Thus if the Einstein condition $R_{ab} = \Lambda g_{ab}$ holds, the Ricci form is self-dual, and in fact is just ΛJ_{ab}.

It is useful when considering Kähler manifolds to introduce the concept of *holomorphic sectional curvature.* We begin by defining the *sectional curvature* on two-planes p at the point x in a Riemannian manifold M by

$$K(p) = R(X, Y, X, Y), \tag{4.13}$$

where X and Y are a pair of orthonormal vectors spanning p, and $R(X, Y, Z, W) = R_{abcd}X^aY^bZ^cW^d$ (Kobayashi and Nomizu 1969). One can easily show that $K(p)$ is independent of the choice of orthonormal basis (X, Y).

The holomorphic sectional curvature of a Kähler manifold is defined to be the sectional curvature on p at $x \in M$ for all planes p invariant by the complex structure $\hat{J}$; i.e. for which if (X, Y) is an orthonormal basis, then so is $(\hat{J}X, \hat{J}Y)$, where $(\hat{J}X)^a = J^a{}_bX^b$. If X is a unit vector in p, and p is a holomorphic two-plane, then $(X, \hat{J}X)$ is always an orthonormal basis for p, and so the holomorphic sectional curvature is given by

$$K_{\mathrm{H}}(p) = R(X, \hat{J}X, X, \hat{J}X). \tag{4.14}$$

The manifold is said to have constant holomorphic sectional curvature if $K_{\mathrm{H}}(p)$ is independent of the choice of two-plane at x. It is then straightforward to show that this implies that M is an Einstein space, $R_{ab} = \Lambda g_{ab}$, with Riemann tensor given by

$$R_{abcd} = \tfrac{1}{6}\Lambda(G_{abcd} + \Sigma_{abcd}), \tag{4.15}$$

where G and Σ are defined in equation (4.9). From this it follows that the Weyl tensor is self-dual, so ${}^{-}C_{abcd} = 0$. One can conversely show that a Kähler manifold with self-dual Weyl tensor satisfies $R = \text{constant}$. If in addition it is Einstein, then it has constant holomorphic sectional curvature.

It is natural then in a general Kähler four-manifold to define a tensor T_{abcd} by

$$T_{abcd} = R_{abcd} - \tfrac{1}{24}R(G_{abcd} + \Sigma_{abcd}), \tag{4.16}$$

from which it follows, since $\|T_{abcd}\|^2 \geqslant 0$, that

$$\|R_{abcd}\|^2 \geqslant \tfrac{1}{3}R^2, \tag{4.17}$$

equality holding if and only if M has constant holomorphic sectional curvature. Substituting this into the expressions in § 3 for χ, τ and C_1^2 gives the result that in an *Einstein–Kähler* manifold,

$$\chi \geqslant 3\tau. \tag{4.18}$$

Once again, this inequality is saturated by the constant holomorphic sectional curvature condition. An example of such a manifold is $P_2(\mathbb{C})$, the complex projective plane, which admits an Einstein–Kähler metric (e.g. Eguchi and Freund 1976, Gibbons and Pope 1978). It has $\chi = 3$, $\tau = 1$.

Finally in this section we consider the circumstances under which a Kähler metric can be regarded as an exact solution of the Einstein–Maxwell equations, with Maxwell field $F = (1/2e)P$. Clearly any Einstein–Kähler solution may be so regarded, since then $P = \Lambda J$, so F not only satisfies Maxwell's equations, but is self-dual and therefore its energy–momentum tensor vanishes. In fact the necessary and sufficient condition turns out to be $R = \text{positive constant}$. To see this, start from the Einstein–Maxwell Lagrangian

$$I = -\frac{1}{2\kappa^2}\int_M (R - 2\Lambda)*1 + \tfrac{1}{4}\int_M F_{ab}F^{ab}*1, \tag{4.19}$$

which implies the classical field equations

$$R = 4\Lambda, \qquad E_{ab} = \kappa^2 T_{ab}, \qquad \mathrm{d}{*}\mathrm{F} = 0, \tag{4.20}$$

where T_{ab} is the (tracefree) electromagnetic stress-tensor

$$T_{ab} = F_{ac}F_{bc} - \tfrac{1}{4}\|F_{cd}\|^2 g_{ab} = 2\,{}^+F_{ac}{}^-F_{bc}. \tag{4.21}$$

Now $P_{ab} = \frac{1}{4}RJ_{ab} + {}^-P_{ab}$, so from (4.11) it follows that

$$E_{ab} = (4/R)\,{}^+P_{ac}{}^-P_{bc}. \tag{4.22}$$

However $F = (1/2e)P$, so (4.20) and (4.21) imply

$$E_{ab} = (\kappa^2/2e^2)\,{}^+P_{ac}{}^-P_{bc}, \tag{4.23}$$

and so equating (4.22) and (4.23) gives

$$2e^2 = \Lambda\kappa^2. \tag{4.24}$$

It remains to check that the third equation in (4.20) is satisfied, $\mathrm{d}{*}F = 0$, or equivalently $\nabla^a F_{ab} = 0$. From (4.11) and the contracted Bianchi identity, it follows that $\nabla^a F_{ab} = (1/4e)(\nabla^a R)J_{ab}$, and so $\mathrm{d}{*}F = 0$ if and only if $R =$ constant, which is already guaranteed by the first equation in (4.20).

To summarise, if one has a Kähler metric with $R = \text{constant} = 4\Lambda$, then one can *choose* to ascribe its Ricci curvature to that implied by the coupled Einstein–Maxwell equations (4.20), with the Maxwell field being $(1/2e)P$. Since condition (4.24) must hold, it follows that Λ must be positive. Note that if the metric is actually Einstein, $R_{ab} = \Lambda g_{ab}$, then (4.22) and (4.23) are both zero, and so (4.24) no longer has to be satisfied.

5. The spectra of Laplacians on Kähler manifolds

In this section we consider the second-order wave operators for fields of various spins, and show how the special properties of Kähler manifolds give rise to certain relations between the eigenfunctions and eigenvalues of the operators. The operators to be considered here are the wave operators for spin-0 and spin-1 fields, for self-dual and anti-self-dual two-forms, the Lichnerowicz operator for spin 2 (i.e. metric perturbations), and the squared Dirac operator for spin $\frac{1}{2}$. A discussion of arbitrary spin operators may be found in Christensen and Duff (1979). The operators for spin 0, spin 1 and for two-forms will just be the Hodge–de Rham operators on zero-, one- and two-forms respectively. The isomorphism between spinors and antiholomorphic forms means that the squared Dirac operator is also of Hodge–de Rham type—in fact the operator $2(\bar\partial + \bar\partial^*)^2$ acting on antiholomorphic forms. The spin-2 operator acts on symmetric tracefree tensors h_{ab}, and it turns out that these may be expressed as some linear combination of terms of the form $\omega_{ac}G_{cb}$, where ω is an anti-self-dual eigenfunction and $G = J$, K or L (see (2.50), (2.51)).

One can summarise the above in table 1.

The problem is thus essentially reduced to looking at Hodge–de Rham operators acting on $\Lambda^{0,0}$, $\Lambda^{0,1}$, $\Lambda^{0,2}$ and $\Lambda^{1,1}$. There is a complication in the case of spin 2 since the two-forms K and L are electrically charged, which means that the anti-self-dual eigenfunctions ω which multiply them must have the opposite charge in order to make h_{ab} electrically neutral. There is also the question of the charge of the spin-$\frac{1}{2}$ eigenfunctions. One might at first think that the natural choice would be uncharged spin-$\frac{1}{2}$ fields, but it turns out to be much more natural to consider spin-$\frac{1}{2}$ fields of charge e

Table 1.

Spin	Corresponding exterior forms
0	$\Lambda^{0,0}$
$\frac{1}{2}$L	$\Lambda^{0,1}$
$\frac{1}{2}$R	$\Lambda^{0,0} \oplus \Lambda^{0,2}$
1	$\Lambda^{0,1} \oplus \Lambda^{1,0}$
Anti-self-dual two-forms	$\Lambda_{\perp}^{1,1}$
Self-dual two-forms	$\Lambda^{0,2} \oplus \Lambda^{2,0} \oplus \Lambda_J^{1,1}$
2	$\Lambda_{\perp}^{1,1} \otimes G$

(the same as the gauge-covariantly constant spinor u_A), which implies that the corresponding antiholomorphic forms are uncharged (see § 2). This choice also has the merit that spinors of charge e can be defined in any Kähler manifold, whereas uncharged spinors can be defined only if the manifold admits a spin structure.

The basic conclusion of this section will be that, provided the condition R = constant holds (except in the case of spin 2, where the stronger Einstein condition $R_{ab} = \Lambda g_{ab}$ must hold), all the eigenfunctions listed in the table can be constructed out of eigenfunctions of certain families of *charged* scalar eigenfunctions, and the eigenvalues of the operators are simply related to the charged scalar eigenvalues.

We begin by introducing the following second-order differential operators:

$$\Delta = (\mathrm{D} + \mathrm{D}^*)^2 = -(*\mathrm{D}*\mathrm{D} + \mathrm{D}*\mathrm{D}*), \tag{5.1}$$

$$\Delta^+ = 2(\mathrm{D}^+ + \mathrm{D}^{+*})^2 = -2(\mathrm{D}^+*\mathrm{D}^-* + *\mathrm{D}^-*\mathrm{D}^+), \tag{5.2}$$

$$\Delta^- = 2(\mathrm{D}^- + \mathrm{D}^{-*})^2 = -2(\mathrm{D}^-*\mathrm{D}^+* + *\mathrm{D}^+*\mathrm{D}^-). \tag{5.3}$$

The first of these is the usual Hodge–de Rham Laplacian, whilst the latter two are the holomorphic and antiholomorphic Laplacians $2\Box$ and $2\bar{\Box}$ discussed by Wells (1979). In each case we have made explicit the gauged derivatives D^+ and D^- (see (2.47)) which reduce to ∂ and $\bar{\partial}$ when acting on uncharged forms. The following properties are easily established:

$$[*, \Delta] = 0, \qquad *\Delta^+ = \Delta^-*, \qquad *\Delta^- = \Delta^+*, \qquad [\mathrm{D}^+, \Delta^+] = [\mathrm{D}^-, \Delta^-] = 0. \tag{5.4}$$

When acting on uncharged forms, the three operators Δ, Δ^+ and Δ^- are all equal in a Kähler manifold, but on charged forms this is no longer in general true. However it turns out that in the case of two-forms, the operators are still equal even when acting on charged forms. Thus for all of the fields to be considered in this section, one can choose to work with whichever of the Laplacians is the most convenient. If one wanted to consider fields of different charges from those to be discussed here, one would have to distinguish between the different Laplacians.

Given a (possibly charged) scalar eigenfunction ϕ one can construct the following forms of higher degree:

$$A\phi \equiv \mathrm{D}^-\phi \quad \text{or} \quad B\phi \equiv *\mathrm{D}^+(L\phi) \qquad \in \Lambda^{0,1}, \tag{5.5}$$

$$J\phi \in \Lambda_J^{1,1}, \qquad L\phi \in \Lambda^{0,2}, \tag{5.6}$$

$$C\phi \equiv \frac{1-*}{2}\mathrm{D}^+\mathrm{D}^-\phi \quad \text{or} \quad E\phi \equiv \mathrm{D}^+*\mathrm{D}^+(L\phi) \quad \text{or} \quad G\phi \equiv \mathrm{D}^-*\mathrm{D}^-(K\phi) \in \Lambda_{\perp}^{1,1} \tag{5.7}$$

where J, K and L are defined by equations (2.50), (2.51) (see § 2 for the definitions of $\Lambda_J^{1,1}$ and $\Lambda_\perp^{1,1}$). It turns out that any eigenfunction of Δ, Δ^+ or Δ^- contained in $\Lambda^{0,1}$, $\Lambda^{0,2}$ or $\Lambda^{1,1}$ can be expressed in terms of these forms, and hence we see from table 1 that any of the higher-spin eigenfunctions being considered here can be so constructed. (Eigenfunctions in $\Lambda^{1,0}$ or $\Lambda^{2,0}$ follow from those in $\Lambda^{0,1}$ or $\Lambda^{0,2}$ by complex conjugation.) Since J is uncharged, K has charge $2e$ and L has charge $-2e$, one has to choose the charge of ϕ so as to produce eigenfunctions of the desired charge.

A straightforward calculation shows that on charged scalar eigenfunctions ϕ_n with charge ne,

$$\Delta^+\phi_n = -\mathrm{D}^a\mathrm{D}_a\phi_n + \tfrac{1}{4}nR\phi_n, \qquad \Delta^-\phi_n = -\mathrm{D}^a\mathrm{D}_a\phi_n - \tfrac{1}{4}nR\phi_n. \tag{5.8}$$

Thus if ϕ_n is an eigenfunction of $-\mathrm{D}^a\mathrm{D}_a$ with eigenvalue λ_n,

$$-\mathrm{D}^a\mathrm{D}_a\phi_n = \lambda_n\phi_n, \tag{5.9}$$

then

$$\Delta^+\phi_n = (\lambda_n + \tfrac{1}{4}nR)\phi_n, \qquad \Delta^-\phi_n = (\lambda_n - \tfrac{1}{4}nR)\phi_n, \tag{5.10}$$

so provided $R = \text{constant}$, ϕ_n is an eigenfunction of Δ^+ and Δ^- also.

To illustrate how the procedure for generating eigenfunctions of higher degree operates, consider the case of antiholomorphic one-forms, given by equation (5.5). For the first possibility one has

$$\Delta^-(A\phi_0) = \Delta^-\mathrm{D}^-\phi_0 = \mathrm{D}^-\Delta^-\phi_0 = \lambda_0(A\phi_0), \tag{5.11}$$

and for the second

$$\Delta^-(B\phi_2) = \Delta^-({*}\mathrm{D}^+(L\phi_2)) = {*}\mathrm{D}^+L\Delta^+\phi_2 = (\lambda_2 + \tfrac{1}{2}R)(B\phi_2). \tag{5.12}$$

Thus $A\phi_0$ and (provided $R = 4\Lambda = \text{constant}$) $B\phi_2$ are $\Lambda^{0,1}$ eigenfunctions, with eigenvalues λ_0 and $\lambda_2 + 2\Lambda$ respectively.

The proof that these $\Lambda^{0,1}$ eigenfunctions are complete proceeds by inverting the above construction and reducing an arbitrary $\Lambda^{0,1}$ eigenfunction down to $\Lambda^{0,0}$ eigenfunctions and then regenerating the original one in $\Lambda^{0,1}$. To do this we introduce the adjoint operators A^*, B^*, defined via the Hodge inner product (2.8), i.e. A^* is defined so that $(\phi, A\psi) = (A^*\phi, \psi)$, etc. One finds that A^* and B^* are given by

$$A^*\eta = {*}\mathrm{D}^+{*}\eta, \qquad B^*\eta = {*}\mathrm{D}^-(K\wedge\eta), \tag{5.13}$$

where $\eta \in \Lambda^{0,1}$. One can check that if η is a $\Lambda^{0,1}$ eigenfunction with eigenvalue μ, $\Delta^-\eta = \mu\eta$, then

$$\eta = -(2/\mu)AA^*\eta - (2\mu)^{-1}BB^*\eta, \tag{5.14}$$

proving that provided $\mu \neq 0$, an arbitrary $\Lambda^{0,1}$ eigenfunction can be expressed in terms of the eigenfunctions of equation (5.5), i.e. $A\phi_0$ and $B\phi_2$ for appropriate ϕ_0 and ϕ_2.

For the case of the self-dual two-forms (5.6), if $R = 4\Lambda$ constant then

$$\Delta^-(J\phi_0) = \lambda_0(J\phi_0), \qquad \Delta^-(L\phi_2) = (\lambda_2 + 2\Lambda)(L\phi_2). \tag{5.15}$$

These, together with the complex conjugate $K\phi_{-2}$ of $L\phi_2$, generate all the self-dual eigenfunctions. The proof in this case is trivial since the relations are purely algebraic.

For the anti-self-dual two-forms (5.7)

$$\begin{aligned} &\Delta^-(C\phi_0) = \lambda_0(C\phi_0), \\ &\Delta^-(E\phi_2) = (\lambda_2 + 2\Lambda)(E\phi_2), \qquad \Delta^-(G\phi_{-2}) = (\lambda_2 + 2\Lambda)(G\phi_{-2}). \end{aligned} \tag{5.16}$$

The adjoints of the operators C, E and G are

$$C^*\omega = *\mathrm{D}^+\mathrm{D}^-\omega,$$
$$E^*\omega = *\mathrm{D}^-(K\wedge *\mathrm{D}^-\omega), \qquad G^*\omega = *\mathrm{D}^+(L\wedge *\mathrm{D}^+\omega), \tag{5.17}$$

acting on $\omega\in\Lambda_\perp^{1,1}$. If ω is an eigenfunction with eigenvalue μ, then $C^*\omega$, $E^*\omega$ and $G^*\omega$ are all scalar eigenfunctions, and the following identity holds:

$$\omega = \mu^{-2}(8CC^*\omega - EE^*\omega - GG^*\omega), \tag{5.18}$$

proving the completeness of the eigenfunctions (5.7) provided $\mu\neq 0$.

The results so far have shown that all (uncharged) $\Lambda^{0,1}$, $\Lambda^{0,2}$ and $\Lambda^{1,1}$ eigenfunctions with non-zero eigenvalues can be constructed from certain (possibly charged) scalar eigenfunctions. Reference to table 1 shows that these cover the cases of spins 0, $\frac{1}{2}$ and 1, and self-dual and anti-self-dual two-forms. Before moving on to discuss the rather more complicated spin-2 case, we consider one further slight subtlety. There can arise certain special cases in which one or more of the operators A, B, C, E or G annihilates the scalar eigenfunction ϕ_n which it is applied to. This occurs if the eigenvalue λ_n of ϕ_n takes certain special values. An obvious example is that the lowest uncharged eigenfunction, $\phi_0=$ constant, is annihilated by all the derivative operators. To investigate this, we note that for each of these operators, the operator followed by its adjoint, applied to a scalar eigenfunction, gives back an (eigenvalue dependent) multiple of the eigenfunction; e.g. $A^*A\phi_0=-\frac{1}{2}\lambda_0\phi_0$. Thus generically, denoting one of the operators by H, one has $H^*H\phi=\alpha\phi$ for some eigenvalue dependent constant α, and so

$$\int_M \bar\phi H^*H\phi *1 = \alpha\int_M |\phi|^2 *1. \tag{5.19}$$

But by construction one can integrate the left-hand side by parts to give

$$\int_M |H\phi|^2 *1 = \alpha\int_M |\phi|^2 *1, \tag{5.20}$$

and so $H\phi=0$ if and only if $\alpha=0$. Some straightforward calculations give the results

$$A^*A\phi_0 = -\tfrac{1}{2}\lambda_0\phi_0, \qquad B^*B\phi_2 = -2(\lambda_2+2\Lambda)\phi_2, \tag{5.21}$$

$$C^*C\phi_0=\tfrac{1}{8}\lambda_0^2\phi_0, \qquad E^*E\phi_2=-(\lambda_2+2\Lambda)^2\phi_2, \qquad G^*G\phi_{-2}=-(\lambda_2+2\Lambda)^2\phi_{-2}. \tag{5.22}$$

Thus as well as $\phi_0=$ constant being annihilated by A and C, if there is a ϕ_2 with $\lambda_2=-2\Lambda$ (which can only happen if $\Lambda<0$, since λ_n cannot be negative), then it is annihilated by B and E.

Turning now to spin 2, the starting point is to note that an arbitrary symmetric tracefree tensor h_{ab} may be expressed as

$$h_{ab} = J_{ac}\overset{0}{\omega}_{cb} + L_{ac}\overset{+}{\omega}_{cb} + K_{ac}\overset{-}{\omega}_{cb}, \tag{5.23}$$

where ω^0, ω^+ and ω^- are certain uniquely determined anti-self-dual two-forms of charge 0, $+2e$ and $-2e$ respectively. In fact, $\overset{0}{\omega}_{ab}=-J_{[a}{}^c h_{b]c}$, $\overset{+}{\omega}_{ab}=-\frac{1}{2}K_{[a}{}^c h_{b]c}$, $\overset{-}{\omega}_{ab}=-\frac{1}{2}L_{[a}{}^c h_{b]c}$, and one can verify by substitution of these into the right-hand side that equation (5.23) always holds.

It turns out to be necessary to impose the Einstein condition $R_{ab}=\Lambda g_{ab}$ in order to construct spin-2 eigenfunctions from scalars, so we will assume this condition from now on. The Lichnerowicz operator Δ_L acting on h_{ab} is then

$$\Delta_L h_{ab} = -\Box h_{ab} - 2R_{acbd}h^{cd} + 2\Lambda h_{ab}. \tag{5.24}$$

Since J, K and L are gauge-covariantly constant, it follows that if h_{ab} is an eigenfunction of Δ_L then ω^0, $\omega^\pm$ are eigenfunctions of Δ^-, and *vice versa*. If $\Delta^-\omega^0 = \mu\omega^0$, then

$$\Delta_L(J_{ac}\overset{0}{\omega}_{cb}) = \mu(J_{ac}\overset{0}{\omega}_{cb}), \tag{5.25}$$

whilst if $\Delta^-\omega^+ = \mu\omega^+$, then

$$\Delta_L(L_{ac}\overset{+}{\omega}_{cb}) = (\mu + 2\Lambda)(L_{ac}\overset{+}{\omega}_{cb}), \tag{5.26}$$

and similarly for the complex conjugate case, $K_{ac}\overset{-}{\omega}_{cb}$. We have already described the construction of the complete set of uncharged anti-self-dual eigenfunctions ω^0, so it just remains to construct those of charge $2e$, ω^+; the ω^- then follow by complex conjugation. This is done in just the same way as in equation (5.7) except that now the scalar eigenfunctions are given charges greater by $2e$ than previously. Thus

$$\omega^+ = C\phi_2 \quad \text{or} \quad E\phi_4 \quad \text{or} \quad G\phi_0. \tag{5.27}$$

Substituting the two-forms (5.27) into Δ^-, one finds after some algebra

$$\Delta^-(C\phi_2) = \lambda_2(C\phi_2), \qquad \Delta^-(E\phi_4) = (\lambda_4 + 6\Lambda)(E\phi_4),$$
$$\Delta^-(G\phi_0) = (\lambda_0 - 2\Lambda)(G\phi_0) \tag{5.28}$$

which when combined with (5.26) shows that these generate spin-2 eigenfunctions with eigenvalues $\lambda_2 + 2\Lambda$, $\lambda_4 + 8\Lambda$ and λ_0 respectively. The complex conjugate eigenfunctions ω^- similarly yield three families of spin-2 eigenfunctions with these eigenvalues also.

The proof of completeness of the spin-2 eigenfunctions is more complicated than for the lower spins. Clearly, since (5.23) is a purely algebraic relationship, it is equivalent to show that the anti-self-dual two-forms are complete, and as the uncharged ones ω^0 have already been shown to be complete, this leaves those of charge $\pm 2e$. A straightforward calculation shows that for ω^+ (of charge $2e$) equation (5.17) becomes

$$\omega^+ = \frac{8}{\mu^2 - 4\Lambda^2}CC^*\omega^+ - \frac{1}{\mu(\mu - 2\Lambda)}EE^*\omega - \frac{1}{\mu(\mu + 2\Lambda)}GG^*\omega, \tag{5.29}$$

where, as before, $\Delta^-\omega^+ = \mu\omega^+$. Thus in general the charge-$2e$ anti-self-dual eigenfunctions obtained from scalars are complete, the possible exceptions being those corresponding to zero modes of Δ^-, or else modes with $\mu = \pm 2\Lambda$. Since Δ^- is a non-negative operator, the cases $\mu = 2\Lambda$ and -2Λ can occur when Λ is positive or negative respectively. Referring to equation (5.26), we see that zero-eigenvalue two-forms generate spin-2 modes with $\Delta_L = 2\Lambda$. These are in fact zero modes of the operator describing the second variation of the Einstein action. It is not clear what is the significance of the apparent breakdown of the eigenfunction generation procedure when $\mu = \pm 2\Lambda$. We will defer further discussion of these points until the next section.

It remains to investigate which scalar eigenfunctions ϕ are annihilated by the operators C, E and G when constructing charge-$2e$ two-forms. The analogue of equations (5.22) in this charged case are

$$C^*C\phi_2 = \tfrac{1}{8}(\lambda_2+2\Lambda)(\lambda_2-2\Lambda)\phi_2,$$
$$E^*E\phi_4 = -(\lambda_4+4\Lambda)(\lambda_4+6\Lambda)\phi_4, \qquad G^*G\phi_0 = -\lambda_0(\lambda_0-2\Lambda)\phi_0, \tag{5.30}$$

and as before the vanishing of the coefficient on the right-hand side of one of these equations implies that the eigenfunction is annihilated by the corresponding operator.

To conclude this section, we give a summary of the results obtained above, and a table of eigenvalues of the various second-order operators discussed. For each spin-$\frac{1}{2}$ (left) and spin-$\frac{1}{2}$ (right) there are two families of eigenfunctions, one constructed from charge-zero scalars ϕ_0 and one from charge-$2e$ scalars ϕ_2; for spin 1 there are four families, two from ϕ_0 and two from $\phi_{\pm 2}$; for each of self-dual and anti-self-dual two-forms there are three families, one from $\phi_{\pm 2}$; and for spin 2 there are nine tracefree families, three from ϕ_0, four from $\phi_{\pm 2}$ and two from $\phi_{\pm 4}$. The eigenvalues of these eigenfunctions are given in terms of the scalar eigenvalues in table 2.

Table 2.

Spin	Eigenfunction	Eigenvalue
0	ϕ_0	λ_0
$\frac{1}{2}$L	$A\phi_0$	λ_0
	$B\phi_2$	$\lambda_2+2\Lambda$
$\frac{1}{2}$R	ϕ_0	λ_0
	$L\phi_2$	$\lambda_2+2\Lambda$
1	$A\phi_0, \overline{A\phi_0}$	λ_0
	$B\phi_2, \overline{B\phi_2}$	$\lambda_2+2\Lambda$
Anti-self-dual	$C\phi_0$	λ_0
	$E\phi_2, G\phi_{-2}$	$\lambda_2+2\Lambda$
Self-dual	$J\phi_0$	λ_0
	$L\phi_2, K\phi_{-2}$	$\lambda_2+2\Lambda$
2	$J\otimes C\phi_0, K\otimes E\phi_0, L\otimes G\phi_0$	λ_0
	$L\otimes C\phi_2, K\otimes C\phi_{-2}, J\otimes E\phi_2, J\otimes G\phi_{-2}$	$\lambda_2+2\Lambda$
	$L\otimes E\phi_4, K\otimes G\phi_{-4}$	$\lambda_4+8\Lambda$

For spin $\frac{1}{2}$ the antiholomorphic exterior forms isomorphic to spin-$\frac{1}{2}$ (L) and spin-$\frac{1}{2}$ (R) are listed (see table 1). For spin 2, the eigenfunctions listed in table 2 are the products of J, K or L with the anti-self-dual eigenfunctions; e.g. $J\otimes C\phi_0$ denotes the spin-2 eigenfunction h_{ab} given by $h_{ab} = J_{ac}(C\phi_0)_{cb}$.

It is interesting to note that in all cases the eigenvalues of eigenfunctions formed from charge-0 scalars are λ_0, from charge $\pm 2e$ scalars are $\lambda_2+2\Lambda$ and from charge $\pm 4e$ scalars are $\lambda_4+8\Lambda$. The special cases in which the eigenfunctions in table 2 in fact vanish may be read off from equations (5.21), (5.22) and (5.30). Otherwise, apart from the case of zero modes and $\Delta_{\mathrm{L}} = \pm 2\Lambda$ spin-2 modes, the eigenfunctions form complete sets, given that the scalar eigenfunctions ϕ_0, $\phi_{\pm 2}$ and $\phi_{\pm 4}$ are complete.

Finally, we remark that all the derivations in this section may be performed instead using the two-component spinor formalism, and working with spinors directly in the case of spin $\frac{1}{2}$ (rather than the isomorphic antiholomorphic exterior form description used in this paper). This was discussed by Pope (1981b), in which it was also shown that this eigenfunction generation procedure is a generalisation of that described in Hawking and Pope (1978a) in the case of half-flat metrics, which are the same as the case of $\Lambda = 0$ Einstein–Kähler metrics (i.e. Ricci-flat Kähler metrics).

6. Zero modes

We have seen from (5.14) and (5.18) that the eigenfunction construction does not work for zero modes for $\Lambda^{0,1}$ and $\Lambda_{\perp}^{1,1}$ forms, and in the case of spin 2 it also apparently fails for certain other special cases too. For $\Lambda^{0,2}$ and $\Lambda_{J}^{1,1}$ on the other hand, the eigenfunctions are algebraically related to scalar eigenfunctions, and so the zero modes are obtainable from scalars in these cases.

By definition, the number of $\Lambda^{0,1}$ zero modes is $h^{0,1}$, the dimension of the cohomology class $H^{0,1}(M, \mathbb{R})$ (see § 3). Furthermore, it follows by complex conjugation that $h^{1,0} = h^{0,1}$, so since $h^{0,1} + h^{1,0} = b_1$,

$$h^{0,1} = h^{1,0} = \tfrac{1}{2} b_1 \tag{6.1}$$

where b_1 is the first Betti number of M. Note that in a Kähler manifold b_1 must therefore be even. If M is simply connected then $b_1 = 0$ and there are no $\Lambda^{0,1}$ zero modes. If M is an Einstein space, the Hodge–de Rham operator (5.1) on one-forms η is

$$\Delta \eta_a = -\Box \eta_a + \Lambda \eta_a, \tag{6.2}$$

so if $\Lambda > 0$ then $\Delta > 0$ and so in this case too there are no $\Lambda^{0,1}$ zero modes.

The $\Lambda_{\perp}^{1,1}$ forms are anti-self-dual two-forms, so the number of zero modes of this type is b_2^-, the anti-self-dual contribution to the second Betti number (see (3.11)):

$$h_{\perp}^{1,1} = b_2^-. \tag{6.3}$$

The $\Lambda^{0,2}$ and $\Lambda_{J}^{1,1}$ zero modes, together with $\Lambda^{2,0}$, the complex conjugate of $\Lambda^{0,2}$, constitute all the self-dual zero modes, and so

$$2h^{0,2} + h_{J}^{1,1} = b_2^+, \tag{6.4}$$

where we have used $h^{2,0} = h^{0,2}$. Now all $\Lambda_{J}^{1,1}$ modes are given by $J\phi_0$ and so from (5.15) it follows that there is just one $\Lambda_{J}^{1,1}$ zero mode, namely J itself, and so

$$h_{J}^{1,1} = 1, \qquad h^{0,2} = \tfrac{1}{2}(b_2^+ - 1). \tag{6.5}$$

Provided $R = 4\Lambda = \text{constant}$, all $\Lambda^{0,2}$ modes are given by $L\phi_2$, and so it follows from (5.15) that $\Lambda^{0,2}$ zero modes are in 1–1 correspondence with $\lambda_2 = -2\Lambda$ charge-$2e$ scalar eigenfunctions. But $\lambda_n \geqslant 0$, so if $\Lambda > 0$ then $h^{0,2} = 0$, and so by (6.5) $b_2^+ = 1$.

It is interesting to note that in an Einstein–Kähler space with $\Lambda > 0$ the arithmetic genus is therefore

$$a = h^{0,0} - h^{0,1} + h^{0,2} = 1 - 0 + 0 = 1, \tag{6.6}$$

and so from (3.15), $\chi + \tau = 4$. Combining this with (3.9), (3.18) and the fact that $C_1^2 > 0$ if $\Lambda > 0$, one arrives at the topological constraints

$$3 \leqslant \chi \leqslant 11, \qquad 1 \geqslant \tau \geqslant -7, \tag{6.7}$$

for Einstein–Kähler spaces with $\Lambda > 0$.

The second variation of the gravitational action $I = -(1/2\kappa^2)\int_M (R-2\Lambda)*1$ around an Einstein background $g_{\mu\nu}$ is (e.g. Christensen and Duff 1980):

$$\delta^2 I = -\frac{1}{4\kappa^2}\int_M h^{\mu\nu}\Delta^{(\Lambda)}h_{\mu\nu}*1, \tag{6.8}$$

where $\delta g_{\mu\nu} = h_{\mu\nu}$ and $h_{\mu\nu}$ is the harmonic gauge

$$\nabla^\mu(h_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}h^\alpha{}_\alpha) = 0. \tag{6.9}$$

The operator $\Delta^{(\Lambda)}$ is given by

$$\Lambda^{(\Lambda)} = \Delta_{\rm L} - 2\Lambda \tag{6.10}$$

where $\Delta_{\rm L}$ is the Lichnerowicz operator (5.24). Equations (5.25) and (5.26) show that zero modes of $\Delta^{(\Lambda)}$ are either of the form $J_{ac}\overset{0}{\omega}_{cb}$ where ω^0 is an uncharged anti-self-dual eigenfunction with eigenvalue 2Λ, or else $L_{ac}\overset{+}{\omega}_{cb}$ or its complex conjugate, where ω^+ is a charge-$2e$ anti-self-dual zero mode.

When $\Lambda<0$, clearly only $\Delta^{(\Lambda)}$ zero modes of the latter type can occur, because of the positivity of the Laplacian Δ, and there are $2N$ of them where N is the number of charge-$2e$ anti-self-dual zero modes. They satisfy the gauge condition (6.9), and correspond to infinitesimal deformations of the Einstein metric which leave the action invariant.

When $\Lambda>0$ the possibility exists that a zero mode of the type $J_{ac}\overset{0}{\omega}_{cb}$ may occur, where $\Delta\omega^0 = 2\Lambda\omega^0$. However such a mode, although a zero mode of $\Delta^{(\Lambda)}$, does not satisfy the gauge condition (6.9) and so does not correspond to a genuine deformation of the metric which leaves the action invariant. As in the case $\Lambda<0$, zero modes of the type $L_{ac}\overset{+}{\omega}_{cb}$ can (in principle) occur.

7. Zeta functions in Einstein–Kähler backgrounds

The relations derived in § 5 can be used to express the zeta function $\zeta(s) = \Sigma_{\lambda>0}\lambda^{-s}$ for the eigenvalues λ of one of the higher-spin operators in terms of the zeta functions constructed from the appropriate scalar eigenvalues. One can then compare the higher-spin zeta function evaluated at $s=0$ with the B_4 coefficient occurring in the heat kernel expansion for the corresponding operator, thereby obtaining an independent check on the eigenvalue relations.

Since $\zeta(0)$ is the regularised number of non-zero eigenvalues, whilst B_4 includes also the number of zero modes, one has to be careful to allow for the zero modes when performing the comparison. One also has to take care to allow for the cases discussed in § 5 in which a scalar eigenfunction is annihilated by one of the spin-raising operators, otherwise the expression derived for $\zeta(0)$ for the higher-spin wave operator will be overcounting by the number of occurrences of this phenomenon. Both these effects lead to B_4 and $\zeta(0)$ disagreeing by a (finite) integer, so for example, even if one does not have information about the number of zero modes, one still obtains a useful check by comparing B_4 and $\zeta(0)$ modulo the integers.

Before computing zeta functions, we first consider the question of those scalar eigenfunctions which are annihilated by the spin-raising operators. Considering first the case of $\Lambda^{0,1}$, equation (5.21) shows that this occurs if $\lambda_0 = 0$ or if $\lambda_2 = -2\Lambda$. There is just one uncharged eigenfunction with $\lambda_0 = 0$, namely $\phi_0 = $ constant. The case

$\lambda_2 = -2\Lambda$ can only occur if $\Lambda < 0$, since $\lambda_n \geqslant 0$. In fact, the lower bound for charged eigenvalues can be sharpened somewhat by considering equations (5.8), with $R = 4\Lambda$. The operators Δ^+, Δ^- are non-negative, so one has the inequality

$$\lambda_n \geqslant |n\Lambda|, \tag{7.1}$$

with equality implying

$$D^+\phi_n = 0 \qquad \text{or} \qquad D^-\phi_n = 0. \tag{7.2}$$

As was shown in § 6, the number of $\lambda_2 = -2\Lambda$ scalar modes is $h^{0,2}$.

The situation is just the same for $\Lambda_\perp^{1,1}$, the anti-self-dual two-forms. Equation (5.22) shows that the method fails in the case of the single $\lambda_0 = 0$ uncharged scalar mode, and the $h^{0,2}$ charge-$2e$ modes with $\lambda_2 = -2\Lambda$. There is no problem for the self-dual two-forms, $\Lambda^{0,2}$, $\Lambda^{2,0}$ and $\Lambda_J^{1,1}$, since these are obtained from scalars by purely algebraic operations.

For spin 2, where we are assuming that $R_{ab} = \Lambda g_{ab}$, equations (5.30) show that scalar eigenfunctions with eigenvalues $\lambda_0 = 0$ or 2Λ, $\lambda_2 = \pm 2\Lambda$, and $\lambda_4 = -4\Lambda$ or -6Λ are annihilated. Of these, the cases $\lambda_2 = \pm 2\Lambda$ and $\lambda_4 = -4\Lambda$ correspond to the inequality (7.1) being saturated. We have already remarked that $\lambda_0 = 0$ occurs once, and $\lambda_2 = -2\Lambda$ occurs $h^{0,2}$ times.

The case $\lambda_0 = 2\Lambda$, which can of course occur only if $\Lambda > 0$, is of interest because such scalar eigenmodes are in 1–1 correspondence with the Killing vectors in the space. To see this, consider first the identity (assuming $R_{ab} = \Lambda g_{ab}$)

$$2\int_M \|\nabla_{(a}V_{b)}\|^2 {*1} = \int_M [V^a(\Delta - 2\Lambda)V_a + (\nabla_a V^a)^2]{*1}. \tag{7.3}$$

This shows that a divergence-free vector eigenfunction of Δ, with eigenvalue 2Λ, is a Killing vector. Now from table 2, bearing in mind that $\lambda_2 \geqslant 2|\Lambda|$, if $\Lambda > 0$ the only vector modes with eigenvalue 2Λ are $A\phi_0$ and $\overline{A\phi_0}$, with ϕ_0 being an uncharged scalar eigenfunction with $\lambda_0 = 2\Lambda$. Now from (5.5) $A\phi_0 = \bar\partial\phi_0$, $\overline{A\phi_0} = \partial\phi_0$, and one can easily show that the only divergence-free combination is $(\bar\partial - \partial)\phi$, which in component notation corresponds to $J_a{}^b\nabla_b\phi_0$. Thus the Killing vectors coincide with these eigenvectors, where $\lambda_0 = 2\Lambda$. It is not clear whether there is any particular significance to be attached to the cases $\lambda_2 = 2\Lambda$ or $\lambda_4 = -4\Lambda$ or -6Λ. In $P_2(\mathbb{C})$, where one can calculate all the scalar eigenvalues explicitly, there is a decuplet of $\lambda_2 = 2\Lambda$ charge-$2e$ scalar modes (Pope 1980).

Turning now to the calculation of zeta functions for the higher-spin operators, one needs to evaluate sums of the form

$$\eta_n(s,k) = \sum_{\lambda_n + k\Lambda > 0} (\lambda_n + k\Lambda)^{-s} \tag{7.4}$$

at $s = 0$ in terms of the zeta functions $\zeta^{(n)}(s) = \Sigma\lambda_n^{-s}$ for the charge-ne scalar operators, in order to obtain expressions for $\zeta(0)$ for higher spins in terms of scalar zeta functions. To do this, one expands (7.4) in descending powers of λ_n:

$$\begin{aligned}\eta_n(s,k) &= \sum \lambda_n^{-s}\left(1 + \frac{k\Lambda}{\lambda_n}\right)^{-s} \\ &= \zeta^{(n)}(s) - k\Lambda s\zeta^{(n)}(s+1) + \tfrac{1}{2}k^2\Lambda^2 s(s+1)\zeta^{(n)}(s+2) + \ldots.\end{aligned} \tag{7.5}$$

Now $\zeta^{(n)}(s)$ has simple poles only, at $s = 0$, 1 and 2 (see e.g. Hawking 1977) so one finds

$$\eta_n(0, k) = \zeta^{(n)}(0) - k\Lambda s\zeta^{(n)}(s+1)|_{s=0} + \tfrac{1}{2}k^2\Lambda^2 s\zeta^{(n)}(s+2)|_{s=0} \tag{7.6}$$

$$= B_4(n) - k\Lambda B_2(n) + \tfrac{1}{2}k^2\Lambda^2 B_0(n) - \varepsilon_n, \tag{7.7}$$

where $B_0(n)$, $B_2(n)$ and $B_4(n)$ are the first three coefficients in the heat kernel expansion for the charge-ne scalar operator. ε_n is an integer which corrects for the overcounting by the number of zero modes. The three cases of interest will be $\eta_0(0, 0)$, for which $\varepsilon_0 = 1$ since $B_4(0)$ includes the $\lambda_0 = 0$ zero mode, $\eta_2(0, 2)$ for which $\varepsilon_2 = h^{0,2}$ since $B_4(2)$ includes the $h^{0,2}$ $\lambda_2 = -2\Lambda$ modes, and $\eta_4(0, 8)$. The last of these arises only in the spin-2 zeta function, where lack of information about the numbers of the special modes discussed earlier will force us to work modulo the integers. It is therefore sufficient for $\eta_4(0, 8)$ merely to note that ε_4 is an integer.

The B_r coefficients for the charged scalar operator $-D^a D_a$ may be found in, for example, Christensen and Duff (1979), where the commutator term $Y_{ab} = [\nabla_a, \nabla_b]$ is modified by the replacement $\nabla_a \to \nabla_a - ienA_a$, and hence $Y_{ab} \to Y_{ab} - ienF_{ab}$. In an Einstein background $F_{ab} = (\Lambda/2e)J_{ab}$, and so

$$B_0(n) = \frac{V}{16\pi^2}, \qquad B_2(n) = \frac{\Lambda V}{24\pi^2}, \qquad B_4(n) = \tfrac{1}{90}\chi + \frac{12 - 5n^2}{480}C_1^2, \tag{7.8}$$

where $V = \int *1$ is the volume of the space, χ is the Euler number, and C_1^2 is the first Chern number. In obtaining (7.8) use has been made of the equation

$$C_1^2 = \Lambda^2 V/2\pi^2, \tag{7.9}$$

which follows from (3.7) when $R_{ab} = \Lambda g_{ab}$. Thus from (7.7),

$$\eta_n(0, k) = \tfrac{1}{480}(30k^2 - 40k - 5n^2 + 12)C_1^2 + \tfrac{1}{90}\chi - \varepsilon_n. \tag{7.10}$$

We can now apply this formula to the various higher-spin operators.

The cases of spin $\frac{1}{2}(L)$ and spin 1 are equivalent, since $S^- \cong \Lambda^{0,1}$ and $\Lambda^1 = \Lambda^{0,1} + \overline{\Lambda^{0,1}}$, so we shall just consider one of them; the spin-1 case. Referring to table 2, the spin-1 zeta function is given by

$$\zeta_1(s) = 2\Sigma\lambda_0^{-s} + 2\Sigma(\lambda_2 + 2\Lambda)^{-s}, \tag{7.11}$$

and so

$$\zeta_1(0) = 2\eta_0(0, 0) + 2\eta_2(0, 2). \tag{7.12}$$

From (7.7), with $\varepsilon_0 = 1$, $\varepsilon_2 = h^{0,2}$,

$$\zeta_1(0) = \tfrac{2}{45}\chi + \tfrac{11}{60}C_1^2 - 2 - 2h^{0,2}. \tag{7.13}$$

This is to be compared with the B_4 coefficient for a spin-1 field in an Einstein background, which may be found in Christensen and Duff (1979). On using the relation (7.9) this becomes

$$B_4(\text{spin 1}) = -\tfrac{11}{90}\chi + \tfrac{1}{60}C_1^2. \tag{7.14}$$

Now we should have $B_4(\text{spin 1}) = \zeta_1(0) + b_1$, since the first Betti number b_1 $(= 2h^{0,1})$ is equal to the number of vector zero modes. From (7.13) one finds

$$\zeta_1(0) + b_1 = \tfrac{2}{45}\chi + \tfrac{11}{60}C_1^2 - 2(1 - h^{0,1} + h^{0,2}), \tag{7.15}$$

and so since $h^{0,0} = b_0 = 1$, the quantity in parentheses is just the arithmetic genus (3.14). On using (3.15) one finds that (7.15) is indeed equal to (7.14).

For Λ^2_+, the self-dual two-forms, one sees from table 2 that the zeta function $\zeta_+(s)$ evaluated at $s=0$ is

$$\zeta_+(0)=\eta_0(0,0)+2\eta_2(0,2)=\tfrac{1}{30}\chi+\tfrac{19}{120}C_1^2-1-2h^{0,2}. \tag{7.16}$$

On the other hand, from Christensen and Duff (1979) the B_4 coefficient for self-dual two-forms in an Einstein–Kähler background is

$$B_4(+)=\tfrac{1}{30}\chi+\tfrac{19}{120}C_1^2. \tag{7.17}$$

In this case the number of zero modes is b_2^+, which by (6.5) is equal to $1+2h^{0,2}$, and indeed adding this to (7.16) does give the required result, (7.17).

For Λ^2_- the non-zero eigenvalues are the same as Λ^2_+, so

$$\zeta_-(0)=\tfrac{1}{30}\chi+\tfrac{19}{120}C_1^2-1-2h^{0,2} \tag{7.18}$$

as for Λ^2_+. From Christensen and Duff (1979)

$$B_4(-)=\tfrac{21}{30}\chi-\tfrac{21}{120}C_1^2. \tag{7.19}$$

The number of anti-self-dual zero modes is $b_2^-=b_2^+-\tau=1+2h^{0,2}-\tau$, and adding this to (7.18) does indeed give (7.19).

For spin 2 we shall merely check the agreement between the zeta function and the B_4 coefficient modulo the integers, since there seems not to be any way of independently determining the number of scalar modes annihilated by the spin raising operators. From table 2, one has for tracefree spin-2 fields

$$\zeta_2(0)=3\eta_0(0,0)+4\eta_2(0,2)+2\eta_4(0,8)=\tfrac{1}{10}\chi+\tfrac{269}{40}C_1^2 \qquad \mathrm{mod}\ \mathbb{Z}. \tag{7.20}$$

On the other hand, from Christensen and Duff (1979) the spin-2 B_4 coefficient is

$$B_4(\text{spin }2)=\tfrac{189}{90}\chi+\tfrac{69}{40}C_1^2. \tag{7.21}$$

The difference $B_4(\text{spin }2)-\zeta_2(0)=2\chi-5C_1^2$ which is indeed an integer. In examples where the eigenvalues can be calculated explicitly, one can check the spin-2 case including the zero modes, and one finds exact agreement. This was done in Pope (1980) for $P_2(\mathbb{C})$; a similar calculation in $S^2\times S^2$ also yields exact agreement.

8. Conclusion

We have seen how the special properties of Kähler manifolds result in certain simplifications of some of the calculations in which one is interested in quantum gravity or quantum field theory in a curved space background; in particular, the second-order wave operators for fields of different spins are related in a special way. These relations depend upon the crucial property of Kähler manifolds that there exists a charged gauge-covariantly constant spinor, by means of which one may associate right- and left-handed spinors with the bundles of even and odd antiholomorphic exterior forms.

An obvious application of some of these ideas would be to the various supergravity models, extending the idea developed by Hawking and Pope (1978a) in the case of half-flat background geometries. Unfortunately the calculations become much more complicated in the present case, because of the fact that it is natural to work with charged fermion fields, coupled to a non-zero background electromagnetic field. One is thus obliged to quantise around a non-trivial Einstein–Maxwell background, and so the operators describing the second variation of the supergravity action are much more complicated than in the pure Einstein case; for example the Einstein–Maxwell part now involves off-diagonal terms coupling metric and Maxwell fluctuations. One might also feel that the necessity of the background Maxwell field makes the model somewhat artificial. On the other hand, in the case of manifolds which do not admit an ordinary spin structure, one is forced to introduce some such background gauge field in order to be able to define spinors consistently at all (e.g. Hawking and Pope 1978b).

In the case of Kähler manifolds with spin structure, one could instead use the methods developed in this paper to construct uncharged fermion eigenfunctions, in which case there would be no need to have a background Maxwell field in the supergravity models. This would be achieved by giving different electric charges to the scalar eigenfunctions used for constructing the fermion eigenfunctions, in order to make the fermions uncharged. However, this would mean that whereas boson fields would all come from scalars of charge 0, $\pm 2e$, $\pm 4e$, the fermions would come from scalars of charge $\pm e$, $\pm 3e$, $\pm 5e$, and so there would appear to be no chance of the boson eigenvalues cancelling against fermion eigenvalues in the functional integral at the one-loop level, in the manner found by Hawking and Pope (1978a). One would at least however have reduced the problem to one involving only scalar eigenfunctions.

Regarded purely as a mathematical tool, the technique of reducing higher-spin fields down to charged scalar fields can be extremely useful for simplifying calculations, as was demonstrated in Pope (1980) in the case of $P_2(\mathbb{C})$. It could also be useful for calculating higher-spin Green functions.

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